Appendix A

Matrix algebra

The mystique surrounding matrix algebra is perhaps due to the texts on the subject requiring a student to 'swallow too much' at one time. It will be found that in order to follow the present text and carry out the necessary computation only a limited knowledge of a few basic definitions is required.

Definition of a matrix

The linear relationship between a set of variables x and b

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 &= b_3 \end{aligned} \tag{A.1}$$

can be written, in a shorthand way, as

$$[A]\{x\} = \{b\} \tag{A.2}$$

or

$$\mathbf{A}\,\mathbf{x} = \mathbf{b} \tag{A.3}$$

where

$$\mathbf{A} \equiv [A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \tag{A.4}$$

$$\mathbf{x} \equiv \{x\} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}$$

$$\mathbf{b} \equiv \{b\} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}$$

The above notation contains within it both the definition of a matrix and of the process of multiplication of two matrices. Matrices are *defined* as 'arrays of numbers' of the type shown in Eq. (A.4). The particular form listing a single column of numbers is often referred to as a vector or column matrix, whereas a matrix with multiple columns and rows is called a rectangular matrix. The multiplication of a matrix by a column vector is *defined* by the equivalence of the left and right sides of Eqs (A.1) and (A.2).

The use of bold characters to define both vectors and matrices will be followed throughout the text, generally lower case letters denoting vectors and capital letters matrices.

If another relationship, using the same a-constants, but a different set of x and b, exists and is written as

$$\begin{aligned} a_{11}x_1' + a_{12}x_2' + a_{13}x_3' + a_{14}x_4' &= b_1' \\ a_{21}x_1' + a_{22}x_2' + a_{23}x_3' + a_{24}x_4' &= b_2' \\ a_{31}x_1' + a_{32}x_2' + a_{33}x_3' + a_{34}x_4' &= b_3' \end{aligned} \tag{A.5}$$

then we could write

$$[A][X] = [B] \qquad \text{or} \qquad \mathbf{AX} = \mathbf{B} \tag{A.6}$$

in which

$$\mathbf{X} \equiv [X] = \begin{bmatrix} x_1, & x_1' \\ x_2, & x_2' \\ x_3, & x_3' \\ x_4, & x_4' \end{bmatrix} \qquad \mathbf{B} \equiv [B] = \begin{bmatrix} b_1, & b_1' \\ b_2, & b_2' \\ b_3, & b_3' \end{bmatrix} \tag{A.7}$$

implying both the statements (A.1) and (A.5) arranged simultaneously as

$$\begin{bmatrix} a_{11}x_1 + \cdots, & a_{11}x_1' + \cdots \\ a_{21}x_1 + \cdots, & a_{21}x_1' + \cdots \\ a_{31}x_1 + \cdots, & a_{31}x_1' + \cdots \end{bmatrix} = \mathbf{B} \equiv [B] = \begin{bmatrix} b_1, & b_1' \\ b_2, & b_2' \\ b_3, & b_3' \end{bmatrix} \tag{A.8}$$

It is seen, incidentally, that matrices can be equal only if each of the individual terms is equal.

The multiplication of full matrices is defined above, and it is obvious that it has a meaning only if the number of columns in **A** is equal to the number of rows in **X** for a relation of the type (A.6).One property that distinguishes matrix multiplication is that, in general,

$$\mathbf{AX} \neq \mathbf{XA}$$

i.e., multiplication of matrices is not commutative as in ordinary algebra.

Matrix addition or subtraction

If relations of the form (A.1) and (A.5) are added then we have

$$\begin{aligned} a_{11}(x_1 + x_1') + a_{12}(x_2 + x_2') + a_{13}(x_3 + x_3') + a_{14}(x_4 + x_4') &= b_1 + b_1' \\ a_{21}(x_1 + x_1') + a_{22}(x_2 + x_2') + a_{23}(x_3 + x_3') + a_{24}(x_4 + x_4') &= b_2 + b_2' \\ a_{31}(x_1 + x_1') + a_{32}(x_2 + x_2') + a_{33}(x_3 + x_3') + a_{34}(x_4 + x_4') &= b_3 + b_3' \end{aligned} \tag{A.9}$$

which will also follow from

$$\mathbf{Ax} + \mathbf{Ax}' = \mathbf{b} + \mathbf{b}'$$

if we define the addition of matrices by simple addition of the individual terms of the array. Clearly this can be done only if the size of the matrices is identical, i.e., for example,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \\ a_{31}+b_{31} & a_{32}+b_{32} \end{bmatrix}$$

or

$$\mathbf{A} + \mathbf{B} = \mathbf{C} \tag{A.10}$$

implies that every term of **C** is equal to the sum of the appropriate terms of **A** and **B**.

Subtraction obviously follows similar rules.

Transpose of a matrix

This is simply a definition for reordering the terms in an array in the following manner:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix} \tag{A.11}$$

and will be indicated by the symbol T as shown.

Its use is not immediately obvious but will be indicated later and can be treated here as a simple prescribed operation.

Inverse of a matrix

If in the relationship (A.2) the matrix **A** is 'square', i.e., it represents the coefficients of simultaneous equations of type (A.1) equal in number to the number of unknowns **x**, then in general it is possible to solve for the unknowns in terms of the known coefficients **b**. This solution can be written as

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} \tag{A.12}$$

in which the matrix $\mathbf{A}^{-1}$ is known as the 'inverse' of the square matrix **A**. Clearly $\mathbf{A}^{-1}$ is also square and of the same size as **A**.

We could obtain (A.12) by multiplying both sides of (A.2) by $\mathbf{A}^{-1}$ and hence

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I} = \mathbf{A}\mathbf{A}^{-1} \tag{A.13}$$

where **I** is an 'identity' matrix having zero on all off-diagonal positions and unity on each of the diagonal positions.

If the equations are 'singular' and have no solution then clearly an inverse does not exist.

A sum of products

In problems of mechanics we often encounter a number of quantities such as force that can be listed as a matrix 'vector':

$$\mathbf{f} = \begin{Bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{Bmatrix} \tag{A.14}$$

These in turn are often associated with the same number of displacements given by another vector, say,

$$\mathbf{a} = \begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{Bmatrix} \tag{A.15}$$

It is known that the work is represented as a sum of products of force and displacement

$$W = \sum_{k=1}^{n} f_k a_k$$

Clearly the transpose becomes useful here as we can write, by the rule of matrix multiplication,

$$W = \begin{bmatrix} f_1 & f_2 \cdots f_n \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{Bmatrix} = \mathbf{f}^{\mathrm{T}}\mathbf{a} = \mathbf{a}^{\mathrm{T}}\mathbf{f} \tag{A.16}$$

Use of this fact is made frequently in this book.

Transpose of a product

An operation that sometimes occurs is that of taking the transpose of a matrix product. It can be left to the reader to prove from previous definitions that

$$(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}} \tag{A.17}$$

Symmetric matrices

In structural problems symmetric matrices are often encountered. If a term of a matrix $\mathbf{A}$ is defined as a_{ij}, then for a symmetric matrix

$$a_{ij} = a_{ji} \qquad \text{or} \qquad \mathbf{A} = \mathbf{A}^{\mathrm{T}}$$

A symmetric matrix must be square. It can be shown that the inverse of a symmetric matrix is also symmetric

$$\mathbf{A}^{-1} = (\mathbf{A}^{-1})^{\mathrm{T}} \equiv \mathbf{A}^{-\mathrm{T}}$$

Partitioning

It is easy to verify that a matrix product **AB** in which for example

$$\mathbf{A} = \left[\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \end{array}\right]$$

$$\mathbf{B} = \left[\begin{array}{cc} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ \hline b_{41} & b_{42} \\ b_{51} & b_{52} \end{array}\right]$$

could be obtained by dividing each matrix into submatrices, indicated by the lines, and applying the rules of matrix multiplication first to each of such submatrices as if it were a scalar number and then carrying out further multiplication in the usual way. Thus, if we write

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}$$

then

$$\mathbf{AB} = \begin{bmatrix} \mathbf{A}_{11}\mathbf{B}_1 & \mathbf{A}_{12}\mathbf{B}_2 \\ \mathbf{A}_{21}\mathbf{B}_1 & \mathbf{A}_{22}\mathbf{B}_2 \end{bmatrix}$$

can be verified as representing the complete product by further multiplication.

The essential feature of partitioning is that the size of subdivisions has to be such as to make the products of the type $\mathbf{A}_{11}\mathbf{B}_1$ meaningful, i.e., the number of columns in $\mathbf{A}_{11}$ must be equal to the number of rows in $\mathbf{B}_1$, etc. If the above definition holds, then all further operations can be conducted on partitioned matrices, treating each partition as if it were a scalar.

It should be noted that any matrix can be multiplied by a scalar (number). Here, obviously, the requirements of equality of appropriate rows and columns no longer apply.

If a symmetric matrix is divided into an equal number of submatrices $\mathbf{A}_{ij}$ in rows and columns then

$$\mathbf{A}_{ij} = \mathbf{A}_{ji}^{\mathrm{T}}$$

The eigenvalue problem

An *eigenvalue* of a symmetric matrix $\mathbf{A}$ of size $n \times n$ is a scalar λ_i, which allows the solution of

$$(\mathbf{A} - \lambda_i \mathbf{I})\boldsymbol{\phi}_i = \mathbf{0} \qquad \text{and} \qquad \det|\mathbf{A} - \lambda_i \mathbf{I}| = 0 \tag{A.18}$$

where $\boldsymbol{\phi}_i$ is called the *eigenvector*.

There are, of course, n such eigenvalues λ_i to each of which corresponds an eigenvector $\boldsymbol{\phi}_i$. Such vectors can be shown to be orthonormal and we write

$$\boldsymbol{\phi}_i^{\mathrm{T}} \boldsymbol{\phi}_j = \delta_{ij} = \begin{cases} 1 & \text{for} \quad i = j \\ 0 & \text{for} \quad i \neq j \end{cases}$$

The full set of eigenvalues and eigenvectors can be written as

$$\boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \qquad \boldsymbol{\Phi} = \begin{bmatrix} \boldsymbol{\phi}_1, & \dots & \boldsymbol{\phi}_n \end{bmatrix}$$

Using these the matrix $\mathbf{A}$ may be written in its *spectral form* by noting from the orthonormality conditions on the eigenvectors that

$$\boldsymbol{\Phi}^{-1} = \boldsymbol{\Phi}^{\mathrm{T}}$$

Then from

$$\mathbf{A}\boldsymbol{\Phi} = \boldsymbol{\Phi}\boldsymbol{\Lambda}$$

it follows immediately that

$$\mathbf{A} = \boldsymbol{\Phi}\boldsymbol{\Lambda}\boldsymbol{\Phi}^{\mathrm{T}} \tag{A.19}$$

The condition number κ (which is related to equation solution roundoff) is defined as

$$\kappa = \frac{|\lambda_{\max}|}{|\lambda_{\min}|} \tag{A.20}$$

Appendix B

Tensor-indicial notation in the approximation of elasticity problems

Introduction

The matrix type of notation used in this volume for the description of tensor quantities such as stresses and strains is compact and we believe easy to understand. However, in a computer program each quantity will often still have to be identified by appropriate indices (see Chapter 20) and the conciseness of matrix notation does not always carry over to the programming steps. Further, many readers are accustomed to the use of indicial-tensor notation which is a standard tool in the study of solid mechanics. For this reason we summarize here the formulation of finite element arrays in an indicial form.

Some advantages of this reformulation from the matrix setting become apparent when evaluation of stiffness arrays for isotropic materials is considered. Here some multiplication operations previously necessary become redundant and the element module programs can be written more economically.

When finite deformation problems in solid mechanics have to be considered the use of indicial notation is almost essential to form many of the arrays needed for the residual and tangent terms.

This appendix adds little new to the discretization ideas – it merely repeats in a different language the results already presented.

Indicial notation: summation convention

A point P in three-dimensional space may be represented in terms of its cartesian coordinates x_a, $a = 1,2,3$. The limits that a can take define its *range*. To define these components we must first establish an oriented orthogonal set of coordinate directions as shown in Fig. B.1. The distance from the origin of the coordinate axes to the point define a position vector $\mathbf{x}$. If along each of the coordinate axes we define the set of unit orthonormal base vectors, $\mathbf{i}_a$, $a = 1,2,3$, which have the property

$$\mathbf{i}_a \cdot \mathbf{i}_b = \delta_{ab} = \begin{cases} 1 \text{ for } a = b \\ 0 \text{ for } a \neq b \end{cases} \tag{B.1}$$

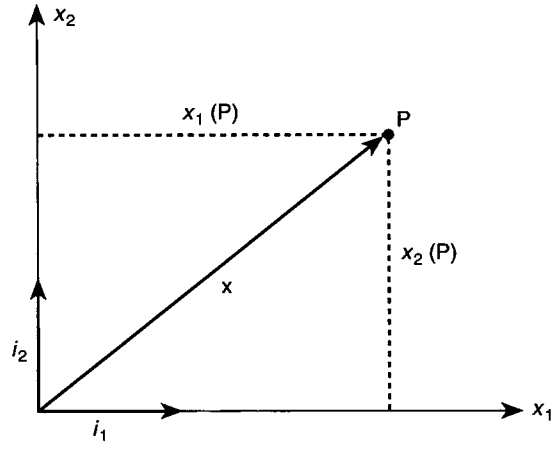


Fig. B.1 Orthogonal axes and a point: cartesian coordinates.

where $(\ )\cdot(\ )$ denotes the vector dot product, the components of the position vector are constructed from the vector dot product

$$x_a = \mathbf{i}_a \cdot \mathbf{x}; \qquad a = 1,2,3 \tag{B.2}$$

From this construction it is easy to observe that the vector $\mathbf{x}$ may be represented as

$$\mathbf{x} = \sum_{a=1}^{3} x_a \mathbf{i}_a \tag{B.3}$$

In dealing with vectors, and later tensors, the form $\mathbf{x}$ is called the *intrinsic* notation of the coordinates and $x_a\mathbf{i}_a$ the indicial form. An intrinsic form is a physical entity which is independent of the coordinate system selected, whereas an indicial form depends on a particular coordinate system.

To simplify notation we adopt the common convention that any index which is repeated in any given term implies a summation over the range of the index. Thus, our shorthand notation for Eq. (B.3) is

$$\mathbf{x} = x_a \mathbf{i}_a = x_1\mathbf{i}_1 + x_2\mathbf{i}_2 + x_3\mathbf{i}_3 \tag{B.4}$$

For two-dimensional problems unless otherwise stated it will be understood that the range of the index is two.

Similarly, we can define the components of the displacement vector $\mathbf{u}$ as

$$\mathbf{u} = u_a \mathbf{i}_a \tag{B.5}$$

Note that the components (u_1, u_2, u_3) replace the components (u, v, w) used throughout most of this volume.

To avoid confusion with nodal quantities to which we previously also attached subscripts we shall simply change their position to a superscript. Thus

$$u_2^j \text{ has the same meaning as } v_j \tag{B.6}$$

used previously, etc.

Derivatives and tensorial relations

In indicial notation the derivative of any quantity with respect to a coordinate component x_a is written compactly as

$$\frac{\partial}{\partial x_a} \equiv (\quad)_{,a} \tag{B.7}$$

Thus we can write the *gradient* of the displacement vector as

$$\frac{\partial u_a}{\partial x_b} \equiv u_{a,b} \qquad a,b = 1,2,3 \tag{B.8}$$

In a cartesian coordinate system the base vectors do not change their magnitude or direction along any coordinate direction. Accordingly their derivatives with respect to any coordinate is zero as indicated in Eq. (B.9):

$$\frac{\partial \mathbf{i}_a}{\partial x_b} = \mathbf{i}_{a,b} = 0 \tag{B.9}$$

Thus in Cartesian co-ordinates, the derivative of the intrinsic displacement $\mathbf{u}$ is given by

$$\mathbf{u}_{,b} = u_{a,b}\mathbf{i}_a + u_a\mathbf{i}_{a,b} = u_{a,b}\mathbf{i}_a \tag{B.10}$$

The collection of all the derivatives defines the *displacement gradient* which we write in intrinsic notation as

$$\nabla\mathbf{u} = u_{a,b}\mathbf{i}_a \otimes \mathbf{i}_b \tag{B.11}$$

The symbol $\otimes$ denotes the *tensor product* between two base vectors and since only two vectors are involved the gradient of the displacement is called *second rank*. The notation used to define a tensor product follows that used in reference 31.

Any second rank intrinsic quantity can be split into symmetric and skew symmetric (antisymmetric) parts as

$$\mathbf{A} = \tfrac{1}{2}[\mathbf{A} + \mathbf{A}^{\mathrm{T}}] + \tfrac{1}{2}[\mathbf{A} - \mathbf{A}^{\mathrm{T}}] = \mathbf{A}^{(s)} + \mathbf{A}^{(a)} \tag{B.12}$$

where $\mathbf{A}$ and its transpose have cartesian components

$$\mathbf{A} = A_{ab}\mathbf{i}_a \otimes \mathbf{i}_b; \qquad \mathbf{A}^{\mathrm{T}} = A_{ba}\mathbf{i}_a \otimes \mathbf{i}_b \tag{B.13}$$

The symmetric part of the displacement gradient defines the (small) strain†

$$\begin{aligned} \boldsymbol{\varepsilon} = \nabla\mathbf{u}^{(s)} &= \tfrac{1}{2}\left[\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{T}}\right] \\ &= \tfrac{1}{2}[u_{a,b} + u_{b,a}]\mathbf{i}_a \otimes \mathbf{i}_b \\ &= \varepsilon_{ab}\,\mathbf{i}_a \otimes \mathbf{i}_b = \varepsilon_{ba}\,\mathbf{i}_a \otimes \mathbf{i}_b \end{aligned} \tag{B.14}$$

and the skew symmetric part gives the (small) rotation

$$\begin{aligned} \boldsymbol{\omega} = \nabla\mathbf{u}^{(a)} &= \tfrac{1}{2}\left[\nabla\mathbf{u} - (\nabla\mathbf{u})^{\mathrm{T}}\right] \\ &= \tfrac{1}{2}[u_{a,b} - u_{b,a}]\mathbf{i}_a \otimes \mathbf{i}_b \\ &= \omega_{ab}\,\mathbf{i}_a \otimes \mathbf{i}_b = -\omega_{ba}\,\mathbf{i}_a \otimes \mathbf{i}_b \end{aligned} \tag{B.15}$$

† Note that this definition is slightly different from that occurring in Chapters 2–6. Now $\varepsilon_{ab} = 1/2\gamma_{ab}$ when $i \neq j$.

The strain expression is analogous to Eq. (2.2). The components ε_{ab} and ω_{ab} may be represented by a matrix as

$$\varepsilon_{ab} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{bmatrix} \tag{B.16}$$

$$\omega_{ab} = \begin{bmatrix} 0 & \omega_{12} & \omega_{13} \\ \omega_{21} & 0 & \omega_{23} \\ \omega_{31} & \omega_{32} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{bmatrix} \tag{B.17}$$

Coordinate transformation

Consider now the representation of the intrinsic coordinates in a system which has different orientation than that given in Fig. B.1. We represent the components in the new system by

$$\mathbf{x} = x'_{a'}\,\mathbf{i}'_{a'} \tag{B.18}$$

Using Eq. (B.2) we can relate the components in the prime system to those in the original system as

$$\begin{aligned} x'_{a'} &= \mathbf{i}'_a \cdot \mathbf{x} \\ &= \mathbf{i}'_a \cdot \mathbf{i}_b x_b \\ &= \Lambda_{a'b} x_b \end{aligned} \tag{B.19}$$

where

$$\Lambda_{a'b} = \mathbf{i}'_{a'} \cdot \mathbf{i}_b = \cos(x'_{a'}, x_b) \tag{B.20}$$

define the direction cosines of the coordinate in a manner similar to that of Eq. (1.25).

Equation (B.19) defines how the cartesian coordinate components transform from one coordinate frame to another. Recall that the summation convention implies

$$x'_{a'} = \Lambda_{a'1} x_1 + \Lambda_{a'2} x_2 + \Lambda_{a'3} x_3 \qquad a' = 1, 2, 3 \tag{B.21}$$

In Eq. (B.19) a' is called a *free index* whereas b is called a *dummy index* since it may be replaced by any other unique index without changing the meaning of the term (note that the notation does not permit an index to appear more than twice in any term). The summation convention will be employed throughout the remainder of this discussion and the reader should ensure that the concept is fully understood before proceeding. Some examples will be given occasionally to illustrate its use.

Using the notion of the direction cosines, Eq. (B.19) may be used to transform any vector with three components. Thus, transformation of the components of the displacement vector is given by

$$u'_{a'} = \Lambda_{a'b} u_b \qquad a'b = 1, 2, 3 \tag{B.22}$$

Indeed we can also use the above to express the transformation for the base vectors since

$$\mathbf{i}'_{a'} = (\mathbf{i}'_{a'} \cdot \mathbf{i}_b)\mathbf{i}_b = \Lambda_{a'b}\mathbf{i}_b \tag{B.23}$$

Similarly, by interchanging the role of the base vectors we obtain

$$\mathbf{i}_b = (\mathbf{i}_b \cdot \mathbf{i}'_{a'})\mathbf{i}'_{a'} = \Lambda_{a'b}\mathbf{i}'_{a'} \tag{B.24}$$

which indicates that the *inverse* of the direction cosine coefficient array is the same as its *transpose*.

The strain transformation follows from the intrinsic form written as

$$\begin{aligned} \boldsymbol{\varepsilon} &= \varepsilon'_{a'b'}\mathbf{i}_{a'} \otimes \mathbf{i}_{b'} \\ &= \varepsilon_{cd}\mathbf{i}_c \otimes \mathbf{i}_d \end{aligned} \tag{B.25}$$

Substitution of the base vectors from Eq. (B.24) into Eq. (B.25) gives

$$\boldsymbol{\varepsilon} = \Lambda_{a'c}\varepsilon_{cd}\Lambda_{b'd}\mathbf{i}_{a'} \otimes \mathbf{i}_{b'} \tag{B.26}$$

Comparing Eq. (B.26) with Eq. (B.25) the components of the strain transform according to the relation

$$\varepsilon'_{a'b'} = \Lambda_{a'c}\varepsilon_{cd}\Lambda_{b'd} \tag{B.27}$$

Variables that transform according to Eq. (B.22) are called *first rank cartesian tensors* whereas quantities that transform according to Eq. (B.27) are called *second rank cartesian tensors*. The use of indicial notation in the context of cartesian coordinates will lead naturally to each mechanics variable being defined in terms of a cartesian tensor of appropriate rank.

Stress may be written in terms of its components σ_{ab} which may be written in matrix form similar to Eq. (B.16)

$$\sigma_{ab} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \qquad a, b = 1, 2, 3 \tag{B.28}$$

In intrinsic form the stress is given by

$$\boldsymbol{\sigma} = \sigma_{ab}\mathbf{i}_a \otimes \mathbf{i}_b \tag{B.29}$$

and, using similar logic as was used for strain, can be shown to transform as a second rank cartesian tensor. The symmetry of the components of stress may be established by summing moments (angular momentum balance) about each of the coordinate axes to obtain

$$\sigma_{ab} = \sigma_{ba} \tag{B.30}$$

Equilibrium and energy

Introducing a body force vector

$$\mathbf{b} = b_a\mathbf{i}_a \tag{B.31}$$

we can write the static equilibrium equations (linear momentum balance) for a differential element as

$$\operatorname{div}\boldsymbol{\sigma} + \mathbf{b} \equiv (\sigma_{ba,b} + b_a)\mathbf{i}_a = \mathbf{0} \tag{B.32}$$

where the repeated index again implies summation over the range of the index, i.e.,

$$\sigma_{ba,b} \equiv \sum_{b=1}^{3} \sigma_{ba,b}$$

$$= \sigma_{1a,1} + \sigma_{2a,2} + \sigma_{3a,3}$$

Note that the free index a must appear in each term for the equation to be meaningful.

As a further example of the summation convention consider an internal energy term

$$W = \sigma_{ab}\varepsilon_{ab} \tag{B.33}$$

This expression implies a double summation; hence summing first on a gives

$$W = \sigma_{1b}\varepsilon_{1b} + \sigma_{2b}\varepsilon_{2b} + \sigma_{3b}\varepsilon_{3b}$$

and then summing on b gives finally

$$\begin{aligned} W = {} & \sigma_{11}\varepsilon_{11} + \sigma_{12}\varepsilon_{12} + \sigma_{13}\varepsilon_{13} \\ & + \sigma_{21}\varepsilon_{21} + \sigma_{22}\varepsilon_{22} + \sigma_{23}\varepsilon_{23} \\ & + \sigma_{31}\varepsilon_{31} + \sigma_{32}\varepsilon_{32} + \sigma_{33}\varepsilon_{33} \end{aligned}$$

We may use symmetry conditions on σ_{ab} and ε_{ab} to reduce the nine terms to six terms. Accordingly,

$$\begin{aligned} W = {} & \sigma_{11}\varepsilon_{11} + \sigma_{22}\varepsilon_{22} + \sigma_{33}\varepsilon_{33} \\ & + 2(\sigma_{12}\varepsilon_{12} + \sigma_{23}\varepsilon_{23} + \sigma_{31}\varepsilon_{31}) \end{aligned} \tag{B.34}$$

Following a similar expansion we can also show the result

$$\sigma_{ab}\omega_{ab} \equiv 0 \tag{B.35}$$

Elastic constitutive equations

For an elastic material the most general linear relationship we can write for components of the stress–strain characterization is

$$\sigma_{ab} = D_{abcd}(\varepsilon_{cd} - \varepsilon^0_{cd}) + \sigma^0_{ab} \tag{B.36}$$

Equation (B.36) is the equivalent of Eq. (2.5) but now written in index notation. We note that the elastic moduli which appear in Eq. (B.36) are components of the fourth rank tensor

$$\mathbf{D} = D_{abcd}\mathbf{i}_a \otimes \mathbf{i}_b \otimes \mathbf{i}_c \otimes \mathbf{i}_d \tag{B.37}$$

The elastic moduli possess the following symmetry conditions

$$D_{abcd} = D_{bacd} = D_{abdc} = D_{cdab} \tag{B.38}$$

the latter, arising from the existence of an internal energy density in the form[2]

$$W(\boldsymbol{\varepsilon}) = \tfrac{1}{2}\varepsilon_{ab}D_{abcd}\varepsilon_{cd} + \varepsilon_{ab}[\sigma^0_{ab} - D_{abcd}\varepsilon^0_{cd}] \tag{B.39}$$

which yields the stress from

$$\sigma_{ab} = \frac{\partial W}{\partial \varepsilon_{ab}} \tag{B.40}$$

By writing the constitutive equation with respect to $x'_{d'}$ and using properties of the base vectors we can deduce the transformation equation for moduli as

$$D'_{a'b'c'd'} = \Lambda_{a'e}\Lambda_{b'f}\Lambda_{c'g}\Lambda_{d'h}D_{efgh} \tag{B.41}$$

A common notation for the intrinsic form of Eq. (B.36) is

$$\boldsymbol{\sigma} = \mathbf{D} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^0) + \boldsymbol{\sigma}^0 \tag{B.42}$$

in which : denotes the double summation (contraction) between the elastic moduli and the strains.

The elastic moduli for an isotropic elastic material may be written in indicial form as

$$D_{abcd} = \lambda\delta_{ab}\delta_{cd} + \mu(\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}) \tag{B.43}$$

where λ, μ are the Lamé constants. An isotropic linear elastic material is always characterized by two independent elastic constants. Instead of the Lamé constants we can use Young's modulus, E, and Poisson's ratio, ν, to characterize the material. The Lamé constants may be deduced from

$$\mu = \frac{E}{2(1+\nu)} \qquad \text{and} \qquad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \tag{B.44}$$

Finite element approximation

If we now introduce the finite element displacement approximation given by Eq. (2.1), using indicial notation we may write for a single element

$$u_a \approx \hat{u}_a = N_i\tilde{u}^i_a \qquad a = 1,2,3; \qquad i = 1,2,\ldots,n \tag{B.45}$$

where n is the total number of nodes on an element. The strain approximation in each element is given by the definition of Eq. (B.14) as

$$\hat{\varepsilon}_{ab} = \tfrac{1}{2}\left[N_{i,b}\tilde{u}^i_a + N_{i,a}\tilde{u}^i_b\right] \tag{B.46}$$

The internal virtual work for an element is given as

$$\delta U^I = \int_{V_e} \delta\boldsymbol{\varepsilon} : \boldsymbol{\sigma}\,\mathrm{d}V = \int_{V_e} \delta\varepsilon_{ab}\sigma_{ab}\,\mathrm{d}V \tag{B.47}$$

Using Eqs (B.46) and (B.47) and noting the symmetries in D_{abcd} we may write the internal virtual work for a linear elastic material as

$$\delta U^I = \delta\tilde{u}^i_a \int_{V_e} N_{i,b}D_{abcd}N_{j,d}\,\mathrm{d}V\tilde{u}^j_c + \delta\tilde{u}^i_a \int_{V_e} N_{i,b}\left(\sigma^0_{ab} - D_{abcd}\varepsilon^0_{cd}\right)\mathrm{d}V \tag{B.48}$$

which replaces the terms obtained in Chapter 2 in indicial notation.

In describing a stiffness coefficient two subscripts have been used previously and the submatrix $\mathbf{K}_{ij}$ implied 2×2 or 3×3 entries for the ij nodal pair, depending on

whether two or three dimensional displacement components were involved. Now the scalar components

$$K_{ab}^{ij} \qquad a,b = 1,2,3; \qquad i,j = 1,2,\ldots,n \tag{B.49}$$

define completely the appropriate stiffness coefficient with *ab* indicating the relative submatrix position (in this case for a three-dimensional displacement).

Note that for a symmetric matrix we have previously required that

$$\mathbf{K}_{ij} = \mathbf{K}_{ji}^{\mathrm{T}} \tag{B.50}$$

In indicial notation the same symmetry is implied if

$$K_{ab}^{ij} = K_{ba}^{ji} \tag{B.51}$$

The stiffness tensor is now defined from Eq. (B.48) as

$$K_{ac}^{ij} = \int_{V_e} N_{i,b} D_{abcd} N_{j,d} \, \mathrm{d}V \tag{B.52}$$

When the elastic properties are constant over the element we may separate the integration from the material constants by defining

$$W_{bd}^{ij} = \int_{V_e} N_{i,b} N_{j,d} \, \mathrm{d}V \tag{B.53}$$

and then perform the summations with the material moduli as

$$K_{ac}^{ij} = W_{bd}^{ij} D_{abcd} \tag{B.54}$$

In the case of isotropy a particularly simple result is obtained

$$K_{ac}^{ij} = \lambda W_{ac}^{ij} + \mu [W_{ca}^{ij} + \delta_{ac} W_{bb}^{ij}] \tag{B.55}$$

which allows the construction of the stiffness to be carried out using fewer arithmetic operations as compared with the use of the matrix form.[3]

Using indicial notation the final equilibrium equations of the system are written as

$$K_{ac}^{ij} u_c^j + f_a^i = 0 \qquad a = 1,2,3 \tag{B.56}$$

and in this scalar form every coefficient is simply identified. The reader can, as a simple exercise, complete the derivation of the force terms due to the initial strain ε_{ab}^0, stress σ_{ab}^0,body force b_a and external traction $\bar{t}_a$.

Indicial notation is at times useful in clarifying individual terms, and this introduction should be helpful as a key to reading some of the current literature.

Relation between indicial and matrix notation

The matrix form used throughout most of this volume can be deduced from the indicial form by a simple transformation between the indices. The relationship between the indices of the second rank tensors and their corresponding matrix form can be performed by an inspection of the ordering in the matrix for stress and its representation shown in Eq. (B.28). In the matrix form the stress was given

in Chapter 6 as

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{22} & \sigma_{33} & \sigma_{12} & \sigma_{23} & \sigma_{31} \end{bmatrix}^{\mathrm{T}} \tag{B.57}$$

This form includes the use of the symmetry of stress components. The mapping of the indices follows that shown in Table B.1.

Table B.1 Mapping between matrix and tensor indices for second rank symmetric tensors

Form	Index number					
Matrix	1	2	3	4	5	6
Tensor	11 *xx*	22 *yy*	33 *zz*	12 & 21 *xy* & *yx*	23 & 32 *yz* & *zy*	31 & 13 *zx* & *xz*

Table B.1 may also be used to perform the map of the material moduli by noting that the components in the energy are associated with the index pairs from the stress and the strain. Accordingly, the moduli transform as

$$D_{1111} \rightarrow D_{11}; \qquad D_{2233} \rightarrow D_{23}; \qquad D_{1231} \rightarrow D_{46}; \quad \text{etc.} \tag{B.58}$$

The symmetry of the stress and strain is embedded in Table B.1 and the existence of an energy function yields the symmetry of the modulus matrix, i.e., $D_{ij} = D_{ji}$.

References

1. P. Chadwick. *Continuum Mechanics*. John Wiley & Sons, New York, 1976.
2. I.S. Sokolnikoff. *The Mathematical Theory of Elasticity*. McGraw-Hill, New York, 2nd edition, 1956.
3. A.K. Gupta and B. Mohraz. A method of computing numerically integrated stiffness matrices. *Internat. J. Num. Meth. Eng.*, **5**, 83–89, 1972.

Appendix C

Basic equations of displacement analysis (Chapter 2)

Displacement

$$\mathbf{u} \approx \hat{\mathbf{u}} = \sum \mathbf{N}_i \mathbf{a}_i = \mathbf{N}\mathbf{a} \tag{C.1}$$

Strain

$$\boldsymbol{\varepsilon} = \mathbf{S}\mathbf{u} \tag{C.2}$$

$$\boldsymbol{\varepsilon} = \sum \mathbf{B}_i \mathbf{a}_i = \mathbf{B}\mathbf{a} \tag{C.3}$$

$$\mathbf{B}_i = \mathbf{S}\mathbf{N}_i \tag{C.4}$$

$$\mathbf{B} = \mathbf{S}\mathbf{N}$$

Stress–strain constitutive relation of linear elasticity

$$\boldsymbol{\sigma} = \mathbf{D}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_0) + \boldsymbol{\sigma}_0 \tag{C.5}$$

Approximate equilibrium equations

$$\mathbf{r} = \mathbf{K}\mathbf{a} + \mathbf{f} \tag{C.6}$$

$$\mathbf{K}_{ij} = \int_V \mathbf{B}_i^{\mathrm{T}} \mathbf{D} \mathbf{B}_j \, \mathrm{d}V \tag{C.7}$$

$$\mathbf{f}_i = -\int_V \mathbf{N}_i^{\mathrm{T}} \mathbf{b} \, \mathrm{d}V - \int_A \mathbf{N}_i^{\mathrm{T}} \bar{\mathbf{t}} \, \mathrm{d}A - \int_V \mathbf{B}_i^{\mathrm{T}} (\mathbf{D}\boldsymbol{\varepsilon}_0 + \boldsymbol{\sigma}_0) \, \mathrm{d}V$$

Appendix D

Some integration formulae for a triangle

Let a triangle be defined in the xy plane by three points (x_i, y_i), (x_j, y_j), (x_m, y_m) with the origin of the coordinates taken at the centroid (or baricentre), i.e.,

$$\frac{x_i + x_j + x_m}{3} = \frac{y_i + y_j + y_m}{3} = 0$$

Then integrating over the triangle area we obtain:

$$\int \mathrm{d}x\,\mathrm{d}y = \frac{1}{2}\begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{vmatrix} = \Delta = \text{area of triangle}$$

$$\int x\,\mathrm{d}x\,\mathrm{d}y = \int y\,\mathrm{d}x\,\mathrm{d}y = 0$$

$$\int x^2\,\mathrm{d}x\,\mathrm{d}y = \frac{\Delta}{12}(x_i^2 + x_j^2 + x_m^2)$$

$$\int y^2\,\mathrm{d}x\,\mathrm{d}y = \frac{\Delta}{12}(y_i^2 + y_j^2 + y_m^2)$$

$$\int xy\,\mathrm{d}x\,\mathrm{d}y = \frac{\Delta}{12}(x_i y_i + x_j y_j + x_m y_m)$$

Appendix E

Some integration formulae for a tetrahedron

Let a tetrahedron be defined in the xyz coordinate system by four points (x_i, y_i, z_i), (x_j, y_j, z_j), (x_m, y_m, z_m), (x_p, y_p, z_p) with the origin of the coordinates taken at the centroid, i.e.,

$$\frac{x_i + x_j + x_m + x_p}{4} = \frac{y_i + y_j + y_m + y_p}{4} = \frac{z_i + z_j + z_m + z_p}{4} = 0$$

Then integrating over the tetrahedron volume gives

$$\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{1}{6}\begin{vmatrix} 1 & x_i & y_i & z_i \\ 1 & x_j & y_j & z_j \\ 1 & x_m & y_m & z_m \\ 1 & x_p & y_p & z_p \end{vmatrix} = V = \text{tetrahedron volume}$$

Provided the order of numbering the nodes is as indicated on Fig. 6.1 then also:

$$\int x\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int y\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = 0$$

$$\int x^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{V}{20}(x_i^2 + x_j^2 + x_m^2 + x_p^2)$$

$$\int y^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{V}{20}(y_i^2 + y_j^2 + y_m^2 + y_p^2)$$

$$\int z^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{V}{20}(z_i^2 + z_j^2 + z_m^2 + z_p^2)$$

$$\int xy\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{V}{20}(x_i y_i + x_j y_j + x_m y_m + x_p y_p)$$

$$\int yz\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{V}{20}(y_i z_i + y_j z_j + y_m z_m + y_p z_p)$$

$$\int zx\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{V}{20}(z_i x_i + z_j x_j + z_m x_m + z_p x_p)$$

Appendix F

Some vector algebra

Some knowledge and understanding of basic vector algebra is needed in dealing with complexities of elements oriented in space as occur in beams, shells, etc. Some of the operations are summarized here.

Vectors (in the geometric sense) can be described by their components along the directions of the x, y, z axes.

Thus, the vector $\mathbf{V}_{01}$ shown in Fig. F.1 can be written as

$$\mathbf{V}_{01} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k} \tag{F.1}$$

in which **i**, **j**, **k** are unit vectors in the direction of the x, y, z axes.

Alternatively, the same vector could be written as

$$\mathbf{V}_{01} = \begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix} \tag{F.2}$$

(now a 'vector' in the matrix sense) in which the components are distinguished by positions in the column.

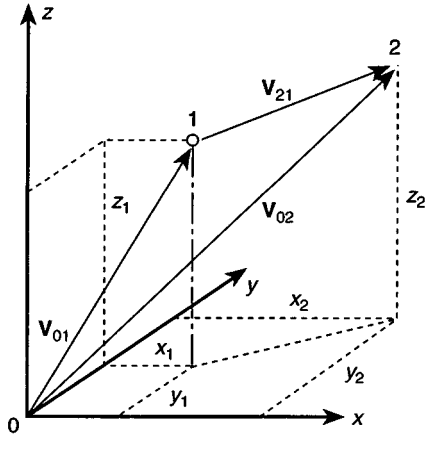


Fig. F.1 Vector addition.

Addition and subtraction

Addition and subtraction is defined by addition and subtraction of components. Thus, for example,

$$\mathbf{V}_{02} - \mathbf{V}_{01} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k} \tag{F.3}$$

The same result is achieved by the definitions of matrix algebra; thus

$$\mathbf{V}_{02} - \mathbf{V}_{01} = \mathbf{V}_{21} = \left\{ \begin{array}{c} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{array} \right\} \tag{F.4}$$

'Scalar' products

The scalar product of two vectors is *defined* as

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} = \sum_{k=1}^{3} a_k b_k \tag{F.5}$$

If

$$\begin{aligned} \mathbf{A} &= a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k} \\ \mathbf{B} &= b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k} \end{aligned} \tag{F.6}$$

then

$$\mathbf{A} \cdot \mathbf{B} = a_x b_x + a_y b_y + a_z b_z \tag{F.7}$$

Using the matrix notation

$$\mathbf{A} = \left\{ \begin{array}{c} a_x \\ a_y \\ a_z \end{array} \right\} \qquad \mathbf{B} = \left\{ \begin{array}{c} b_x \\ b_y \\ b_z \end{array} \right\} \tag{F.8}$$

the scalar product becomes

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{A}^{\mathrm{T}}\mathbf{B} = \mathbf{B}^{\mathrm{T}}\mathbf{A} \tag{F.9}$$

Length of vector

The length of the vector $\mathbf{V}_{21}$ is given, purely geometrically, as

$$l_{21} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{F.10}$$

or in terms of matrix algebra as

$$l_{21} = \sqrt{\mathbf{V}_{21} \cdot \mathbf{V}_{21}} = \sqrt{\mathbf{V}_{21}^{\mathrm{T}}\mathbf{V}_{21}} \tag{F.11}$$

Direction cosines

Direction cosines of a vector are simply given from the definition of the projected component of lengths as (Fig. F.1)

$$\cos\alpha_x = \Lambda_{vx} = \frac{x_2 - x_1}{l_{21}} = \frac{\mathbf{V}\cdot\mathbf{i}}{l_{21}}, \quad \text{etc.} \tag{F.12}$$

The scalar product may also be written as (Fig. F.2)

$$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A} = l_a l_b \cos\gamma \tag{F.13}$$

where γ is the angle between the two vectors **A** and **B** and l_a and l_b are their lengths, respectively.

'Vector' or cross product

Another product of vectors is defined as a vector oriented normally to the plane given by two vectors and equal in magnitude to the product of the length of the two vectors multiplied by the sine of the angle between them. Further, the direction of the normal vector follows the right-hand rule as shown in Fig. F.2 in which

$$\mathbf{A}\times\mathbf{B} = \mathbf{C} \tag{F.14}$$

is shown.

Thus, from the right-hand rule, we have

$$\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A} \tag{F.15}$$

It is worth noting that the magnitude (or length) of **C** is equal to the area of the parallelogram shown in Fig. F.2.

Using the definition of Eq. (F.6) and noting that

$$\begin{aligned} &\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = 0 \\ &\mathbf{i}\times\mathbf{j} = \mathbf{k} \qquad \mathbf{j}\times\mathbf{k} = \mathbf{i} \qquad \mathbf{k}\times\mathbf{i} = \mathbf{j} \end{aligned} \tag{F.16}$$

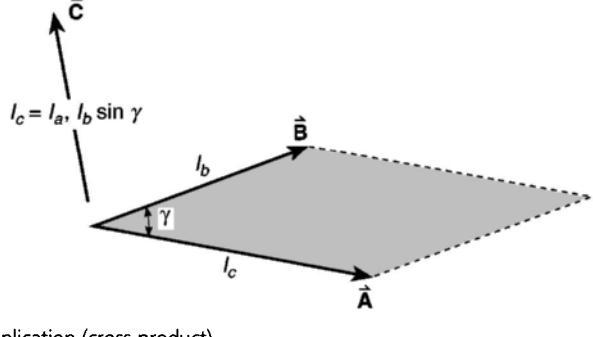


Fig. F.2 Vector multiplication (cross product).

we have

$$\mathbf{A} \times \mathbf{B} = \det \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$
$$= (a_y b_z - a_z b_y)\mathbf{i} + (a_z b_x - a_x b_z)\mathbf{j} + (a_x b_y - a_y b_x)\mathbf{k}$$

There is no simple counterpart in matrix algebra but we can use the above to define the vector $\mathbf{C}$.†

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = \begin{Bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{Bmatrix} \tag{F.17}$$

The vector product will be found particularly useful when the problem of erecting a normal direction to a surface is considered.

Elements of area and volume

If ξ and η are curvilinear coordinates, then the following vectors in the two-dimensional plane

$$\mathrm{d}\boldsymbol{\xi} = \begin{Bmatrix} \dfrac{\partial x}{\partial \xi} \\ \dfrac{\partial y}{\partial \xi} \end{Bmatrix} \mathrm{d}\xi \qquad \mathrm{d}\boldsymbol{\eta} = \begin{Bmatrix} \dfrac{\partial x}{\partial \eta} \\ \dfrac{\partial y}{\partial \eta} \end{Bmatrix} \mathrm{d}\eta \tag{F.18}$$

defined from the relationship between the cartesian and curvilinear coordinates, are vectors directed tangentially to the ξ and η equal-constant contours, respectively. As the *length* of the vector resulting from a cross product of $\mathrm{d}\boldsymbol{\xi} \times \mathrm{d}\boldsymbol{\eta}$ is equal to the area of the elementary parallelogram we can write

$$\mathrm{d}(\mathit{area}) = \det \begin{vmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta} \\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta} \end{vmatrix} \mathrm{d}\xi\, \mathrm{d}\eta \tag{F.19}$$

by Eq. (F.17).

† If we rewrite $\mathbf{A}$ as a skew symmetric matrix

$$\hat{\mathbf{A}} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

then an alternative representation of the vector product in matrix form is $\mathbf{C} = \hat{\mathbf{A}}\mathbf{B}$.

Similarly, if we have three curvilinear coordinates ξ, η, ζ in cartesian space, the 'triple' or box product defines a differential volume

$$\mathrm{d}(vol) = (\mathrm{d}\boldsymbol{\xi} \times \mathrm{d}\boldsymbol{\eta}) \cdot \mathrm{d}\boldsymbol{\zeta} = \det \begin{vmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta} & \dfrac{\partial x}{\partial \zeta} \\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta} & \dfrac{\partial y}{\partial \zeta} \\ \dfrac{\partial z}{\partial \xi} & \dfrac{\partial z}{\partial \eta} & \dfrac{\partial z}{\partial \zeta} \end{vmatrix} \mathrm{d}\xi \, \mathrm{d}\eta \, \mathrm{d}\zeta \tag{F.20}$$

This follows simply from the geometry. The bracketed product, by definition,forms a vector whose length is equal to the parallelogram area with sides tangent to two of the coordinates. The second scalar multiplication by a length and the cosine of the angle between that length and the normal to the parallelogram establishes a differential volume element.

The above equations serve in changing the variables in surface and volume integrals.

Appendix G

Integration by parts in two or three dimensions (Green's theorem)

Consider the integration by parts of the following two-dimensional expression

$$\iint_{\Omega} \phi \frac{\partial \psi}{\partial x} \, dx \, dy \tag{G.1}$$

Integrating first with respect to x and using the well-known relation for integration by parts in one-dimension

$$\int_{x_L}^{x_R} u \, dv = -\int_{x_L}^{x_R} v \, du + (uv)_{x=x_R} - (uv)_{x=x_L} \tag{G.2}$$

we have, using the symbols of Fig. G.1

$$\iint_{\Omega} \phi \frac{\partial \psi}{\partial x} \, dx \, dy = -\iint_{\Omega} \frac{\partial \phi}{\partial x} \psi \, dx \, dy + \int_{y_B}^{y_T} \left[(\phi \psi)_{x=x_R} - (\phi \psi)_{x=x_L} \right] dy \tag{G.3}$$

If we now consider a direct segment of the boundary $d\Gamma$ on the right-hand boundary, we note that

$$dy = n_x \, d\Gamma \tag{G.4}$$

where n_x is the direction cosine between the outward normal and the x direction. Similarly on the left-hand section we have

$$dy = -n_x \, d\Gamma \tag{G.5}$$

The final term of Eq. (G.3) can thus be expressed as the integral taken around an anticlockwise direction of the complete closed boundary:

$$\oint_{\Gamma} \phi \psi n_x \, d\Gamma \tag{G.6}$$

If several closed contours are encountered this integration has to be taken around each such countour. The general expression in all cases is

$$\iint_{\Omega} \phi \frac{\partial \psi}{\partial x} \, dx \, dy = -\iint_{\Omega} \frac{\partial \phi}{\partial x} \psi \, dx \, dy + \oint_{\Gamma} \phi \psi n_x \, d\Gamma \tag{G.7}$$

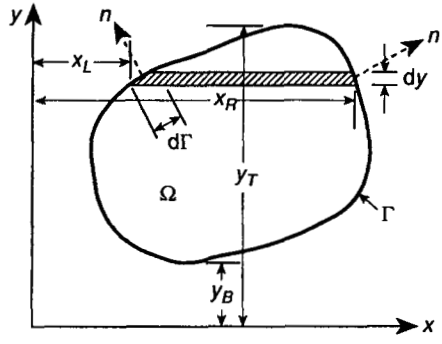


Fig. G.1 Definitions for integrations in two-dimensions.

Similarly, if differentiation in the y direction arises we can write

$$\iint_{\Omega} \phi \frac{\partial \psi}{\partial y} \, dx \, dy = - \iint_{\Omega} \frac{\partial \phi}{\partial y} \psi \, dx \, dy + \oint_{\Gamma} \phi \psi n_y \, d\Gamma \tag{G.8}$$

where n_y is the direction cosine between the outward normal and the y axis.

In three dimensions by an identical procedure we can write

$$\iiint_{\Omega} \phi \frac{\partial \psi}{\partial y} \, dx \, dy \, dz = - \iiint_{\Omega} \frac{\partial \phi}{\partial y} \psi \, dx \, dy \, dz + \oint_{\Gamma} \phi \psi n_y \, d\Gamma \tag{G.9}$$

where $d\Gamma$ becomes the element of the surface area and the last integral is taken over the whole surface. A similar expression holds for derivatives like Eq. (G.9) in x and z.

Appendix H

Solutions exact at nodes

The finite element solution of ordinary differential equations may be made exact at the interelement nodes by a proper choice of the *weighting* function in the weak (Galerkin) form. To be more specific, let us consider the set of ordinary differential equations given by

$$\mathbf{A}(\mathbf{u}) + \mathbf{f}(x) = \mathbf{0} \tag{H.1}$$

where $\mathbf{u}$ is the set of dependent variables which are functions of the single independent variable 'x' and $\mathbf{f}$ is a vector of specified load functions. The weak form of this set of differential equations is given by

$$\int_{x_L}^{x_R} \mathbf{v}^{\mathrm{T}}[\mathbf{A}(\mathbf{u}) + \mathbf{f}]\,\mathrm{d}x = 0 \tag{H.2}$$

The weak form may be integrated by parts to remove all the derivatives from $\mathbf{u}$ and place them on $\mathbf{v}$. The result of this step may be expressed as

$$\int_{x_L}^{x_R} [\mathbf{u}^{\mathrm{T}}\mathbf{A}^*(\mathbf{v}) + \mathbf{v}^{\mathrm{T}}\mathbf{f}]\,\mathrm{d}x + [\mathbf{B}^*(\mathbf{v})]^{\mathrm{T}}\mathbf{B}(\mathbf{u})\Big|_{x_L}^{x_R} = 0 \tag{H.3}$$

where $\mathbf{A}^*(\mathbf{v})$ is the *adjoint differential equation* and $\mathbf{B}^*(\mathbf{v})$ and $\mathbf{B}(\mathbf{u})$ are terms on the boundary resulting from integration by parts.

If we can find the general integral to the homogeneous adjoint differential equation

$$\mathbf{A}^*(\mathbf{v}) = \mathbf{0} \tag{H.4}$$

then the weak form of the problem reduces to

$$\int_{x_L}^{x_R} \mathbf{v}^{\mathrm{T}}\mathbf{f}\,\mathrm{d}x + [\mathbf{B}^*(\mathbf{v})]^{\mathrm{T}}\mathbf{B}(\mathbf{u})\Big|_{x_L}^{x_R} = 0 \tag{H.5}$$

The first term is merely an expression to generate equivalent forces from the solution to the adjoint equation and the last term is used to construct the residual equation for the problem. If the differential equation is linear these lead to a residual which depends linearly on the values of $\mathbf{u}$ at the ends x_L and x_R. If we now let these be the location of the end nodes of a typical element we immediately find an expression to generate a stiffness matrix. Since in this process we have never had to construct an *approximation* for the dependent variables $\mathbf{u}$ it is immediately evident that at the end

points the discrete values of the exact solution must coincide with any admissible approximation we choose. Thus, we always obtain exact solutions at these points.

If we consider that all values of the forcing function are contained in f (i.e., no point loads at nodes), the terms in $\mathbf{B}(\mathbf{u})$ must be continuous between adjacent elements. At the boundaries the terms in $\mathbf{B}(\mathbf{u})$ include a flux term as well as displacements.

As an example problem, consider the single differential equation

$$\frac{d^2u}{dx^2} + P\frac{du}{dx} + f = 0 \tag{H.6}$$

with the associated weak form

$$\int_{x_L}^{x_R} v\left[\frac{d^2u}{dx^2} + P\frac{du}{dx} + f\right] dx = 0 \tag{H.7}$$

After integration by parts the weak form becomes

$$\int_{x_L}^{x_R}\left[u\left(\frac{d^2v}{dx^2} - P\frac{dv}{dx}\right) + vf\right] dx + \left[v\left(\frac{du}{dx} + Pu\right) - \frac{dv}{dx}u\right]_{x_L}^{x_R} = 0 \tag{H.8}$$

The adjoint differential equation is given by

$$a^*(v) = \frac{d^2v}{dx^2} - P\frac{dv}{dx} = 0 \tag{H.9}$$

and the boundary terms by

$$\mathbf{B}^*(v) = \begin{Bmatrix} v \\ -\dfrac{dv}{dx} \end{Bmatrix} \tag{H.10}$$

and

$$\mathbf{B}(u) = \begin{Bmatrix} \dfrac{du}{dx} + Pu \\ u \end{Bmatrix} \tag{H.11}$$

For the above example two cases may be identified:

1. P zero, where the adjoint differential equation is identical to the homogeneous equation in which case the problem is called *self-adjoint*.
2. P non-zero, where we then have the *non-self-adjoint* problem.

The finite element solution for these two cases is often quite different. In the first case an equivalent variational theorem exists, whereas for the second case no such theorem exists.†

In the first case the solution to the adjoint equation is given by

$$v = Ax + B \tag{H.12}$$

which may be written as conventional linear shape functions in each element as

$$N_L = \frac{x_R - x}{x_R - x_L} \qquad N_R = \frac{x - x_L}{x_R - x_L} \tag{H.13}$$

† An integrating factor may often be introduced to make the weak form generate a self-adjoint problem; however, the approximation problem will remain the same. See Sec. 3.9.2.

Thus, for linear shape functions in each element used as the weighting function the interelement nodal displacements for u will always be exact (e.g., see Fig. 3.4) irrespective of the interpolation used for u.

For the second case the exact solution to the adjoint equation is

$$v = Ae^{Px} + B = Az + B \tag{H.14}$$

This yields the shape functions for the weighting function

$$N_L = \frac{z_R - z}{z_R - z_L} \qquad N_R = \frac{z - z_L}{z_R - z_L} \tag{H.15}$$

which when used in the weak form again yield exact answers at the interelement nodes.

After constructing exact nodal solutions for u, exact solutions for the flux at the interelement nodes can also be obtained from the weak form for each element. The above process was first given by Tong for self-adjoint differential equations.†

† P. Tong. Exact solutions of certain problems by the finite element method. *J. AIAA*, 7, 179–80, 1969.

Appendix I

Matrix diagonalization or lumping

Some of the algorithms discussed in this volume become more efficient if one of the global matrices can be diagonalized (also called 'lumped' by many engineers). For example, the solution of some mixed and transient problems are more efficient if a global matrix to be inverted (or equations solved) is diagonal [see Chapter 12, Eq. (12.95) and Chapter 17, Sec. 17.2.4 and 17.4.2]. Engineers have persisted with purely physical concepts of lumping; however, there is clearly a need for devising a systematic and mathematically acceptable procedure for such lumping.

We shall define the matrix to be considered as

$$\mathbf{A} = \int_{\Omega} \mathbf{N}^{\mathrm{T}} \mathbf{c} \mathbf{N} \, \mathrm{d}\Omega \tag{I.1}$$

where $\mathbf{c}$ is a matrix with small dimension. Often $\mathbf{c}$ is a diagonal matrix (e.g., in mass or simple least square problems $\mathbf{c}$ is an identity matrix times some scalar).When $\mathbf{A}$ is computed exactly it has full rank and is not diagonal – this is called the *consistent* form of $\mathbf{A}$ since it is computed consistently with the other terms in the finite element model. The diagonalized form is defined with respect to 'nodes' or the shape functions, e.g., $\mathbf{N}_i = N_i \mathbf{I}$; hence, the matrix will have small diagonal blocks, each with the maximum dimension of c. Only when $\mathbf{c}$ is diagonal can the matrix $\mathbf{A}$ be completely diagonalized. Four basic lines of argument may be followed in constructing a diagonal form.

The first procedure is to use different shape functions to approximate each term in the finite element discretization. For the $\mathbf{A}$ matrix we use substitute shape functions $\bar{\mathbf{N}}_i$ for the lumping process. No derivatives exist in the definition of $\mathbf{A}$, hence, for this term the shape functions may be piecewise continuous within and between elements and still lead to an acceptable approximation. If the shape functions used to define $\mathbf{A}$ are piecewise constants, such that $\bar{\mathbf{N}}_i$ in a certain part of the element surrounding the node i and zero elsewhere, and such parts are not overlapping or disjoint, then clearly the matrix of Eq. (I.1) becomes nodally diagonal as

$$\int_{\Omega} \bar{\mathbf{N}}_i^{\mathrm{T}} \mathbf{c} \bar{\mathbf{N}}_j \, \mathrm{d}\Omega = \begin{cases} \displaystyle\int_{\Omega_i} \mathbf{c} \, \mathrm{d}\Omega & i = j \\ 0 & i \neq j \end{cases} \tag{I.2}$$

Such an approximation with different shape functions is permissible since the usual finite element criteria of integrability and completeness are satisfied. We can verify

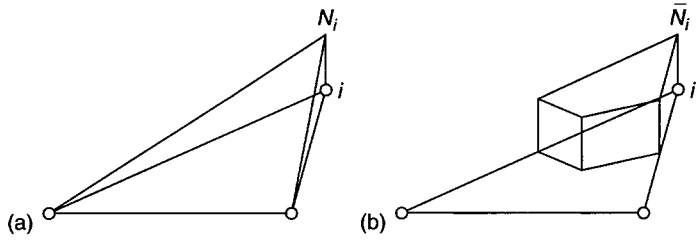


Fig. I.1 (a) Linear and (b) piecewise constant shape functions for a triangle.

this using a patch test to show that consistency is still maintained in the approximation. The functions selected need only satisfy the condition

$$\bar{\mathbf{N}}_i = \bar{N}_i \mathbf{I} \qquad \text{with} \qquad \sum_i \bar{N}_i = 1 \tag{I.3}$$

for all points in the element and this also maintains a partition of unity property in all of Ω. In Fig. I.1 we show the functions N_i and $\bar{N}_i$ for a triangular element.

The second method to diagonalize a matrix is to note that condition (I.1) is simply a requirement that ensures conservation of the quantity $\mathbf{c}$ over the element. For structural dynamics applications this is the conservation of mass at the element level. Accordingly, it has been noted that any lumping that preserves the integral of $\mathbf{c}$ on the element will lead to convergent results, although the rate of convergence may be lower than with use of a consistent $\mathbf{A}$. Many alternatives have been proposed based upon this method. The earliest procedures performed the diagonalization using physical intuition only. Later alternative algorithms were proposed. One suggestion, often called a 'row sum' method, is to compute the diagonal matrix from

$$\mathbf{A}_{ij} = \begin{cases} \sum_k \int_{\Omega_i} \mathbf{N}_i^{\mathrm{T}} \mathbf{c} \mathbf{N}_k \, \mathrm{d}\Omega & i = j \\ \mathbf{0} & i \neq j \end{cases} \tag{I.4}$$

This simplifies to

$$\mathbf{A}_{ij} = \begin{cases} \int_{\Omega_i} \mathbf{N}_i^{\mathrm{T}} \mathbf{c} \, \mathrm{d}\Omega & i = j \\ \mathbf{0} & i \neq j \end{cases} \tag{I.5}$$

since the sum of the shape functions is unity. This algorithm makes sense only when the degrees of freedom of the problem all have the same physical interpretation. An alternative is to scale the diagonals of the consistent mass to satisfy the conservation requirement. In this case the diagonal matrix is deduced from

$$\mathbf{A}_{ij} = \begin{cases} a \int_{\Omega_i} \mathbf{N}_i^{\mathrm{T}} \mathbf{c} \mathbf{N}_i \, \mathrm{d}\Omega & i = j \\ \mathbf{0} & i \neq j \end{cases} \tag{I.6}$$

where a is selected so that

$$\sum_i \mathbf{A}_{ii} = \int_\Omega \mathbf{c} \, \mathrm{d}\Omega \tag{I.7}$$

The third procedure uses numerical integration to obtain a diagonal array without apparently introducing additional shape functions. Use of numerical integration to evaluate the **A** matrix of Eq. (I.1) yields a typical term in the summation form (following Chapter 9)

$$\mathbf{A}_{ij} = \int_{\Omega} \mathbf{N}_i^{\mathrm{T}} \mathbf{c} \mathbf{N}_j \, \mathrm{d}\Omega = \sum_q (\mathbf{N}_i^{\mathrm{T}} \mathbf{c} \mathbf{N}_j)_{\xi_q} J_q W_q \tag{I.8}$$

where ξ_q refers to the quadrature point at which the integrand is evaluated, J is the jacobian volume transformation at the same point and W_q gives the appropriate quadrature weight.

If the quadrature points for the numerical integration are located at nodes then (for standard shape functions) by Eq. (I.3) the diagonal matrix is

$$\mathbf{A}_{ij} = \begin{cases} \mathbf{c} J_i W_i & i = j \\ \mathbf{0} & i \neq j \end{cases} \tag{I.9}$$

where J_i is the jacobian and W_i is the quadrature weight at node i.

Appropriate weighting values may be deduced by requiring the quadrature formula to exactly integrate particular polynomials in the natural coordinate system. In general the quadrature should integrate a polynomial of the highest complete order in the shape functions. Thus, for four-noded quadrilateral elements, linear functions should be exactly integrated. Integrating additional terms may lead to improved accuracy but is not required. Indeed, only conservation of **c** is required.

For low-order elements, symmetry arguments may be used to lump the matrix. It is, for instance, obvious that in a simple triangular element little improvement can be obtained by any other lumping than the simple one in which the total **c** is distributed in three equal parts. For an eight-noded two-dimensional isoparametric element no such obvious procedure is available. In Fig. I.2 we show the case of rectangular

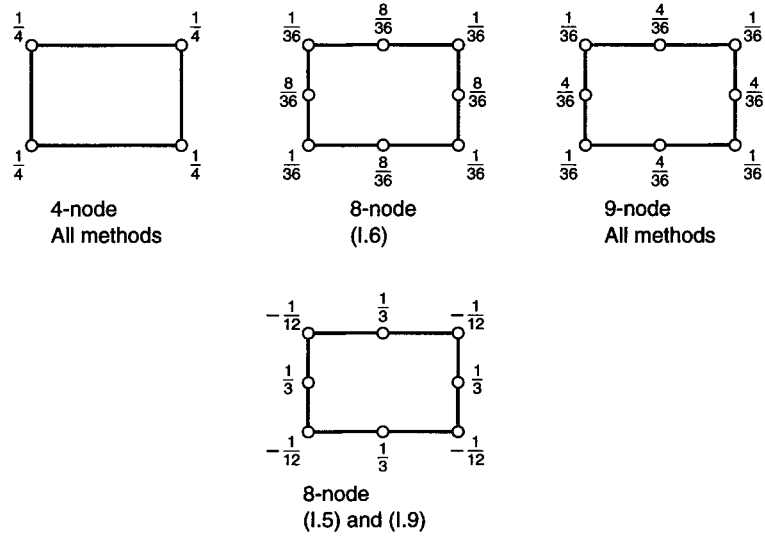


Fig. I.2 Diagonalization of rectangular elements by three methods.

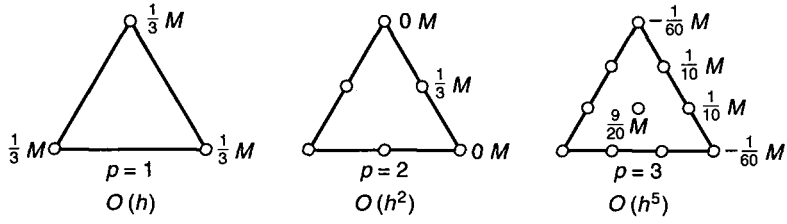


Fig. I.3 Diagonalization of triangular elements by quadrature.

elements of four-, eight-, and nine-noded type and lumping by Eqs (I.5), (I.6) and (I.9).

It is noted that for the eight-noded element some of the lumped quantities are negative when Eq. (I.5) or Eq. (I.9) is used. These will have some adverse effects in certain algorithms (e.g., time-stepping schemes to integrate transient problems) and preclude their use. In Fig. I.3 we show some lumped matrices for triangular elements computed by quadrature (i.e., Eq. (I.9)). It is noted here that the cubic element has negative terms while the quadratic element has zero terms. The zero terms are particularly difficult to handle as the resulting diagonal matrix **A** no longer has full rank and thus may not be inverted.

Another aspect of lumping is the performance of the element when distorted from its parent element shape. For example, as a rectangular element is distorted and approaches a triangular shape it is desirable to have the limit triangular shape case behave appropriately. In the case of a four-noded rectangular element the lumped matrix for all three procedures gives the same answer. However if the element is distorted by a transformation defined by one parameter f as shown in Fig. I.4 then the three lumping procedures discussed so far give different answers. The jacobian transformation is given by

$$J = ab(1-f) \tag{I.10}$$

and **c** is here taken as the identity matrix.

The form (I.5) gives

$$\mathbf{A}_{ii} = \begin{cases} ab(1-f/3) & \text{at top nodes} \\ ab(1+f/3) & \text{at bottom nodes} \end{cases} \tag{I.11}$$

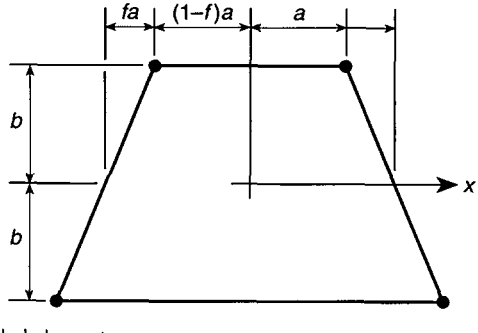


Fig. I.4 Distorted four-noded element.

the form (I.6) gives

$$\mathbf{A}_{ii} = \begin{cases} ab(1-f/2) & \text{at top nodes} \\ ab(1+f/2) & \text{at bottom nodes} \end{cases} \tag{I.12}$$

and the quadrature form (I.9) yields

$$\mathbf{A}_{ii} = \begin{cases} ab(1-f) & \text{at top nodes} \\ ab(1+f) & \text{at bottom nodes} \end{cases} \tag{I.13}$$

The four-noded element has the property that a triangle may be defined by coalescing two nodes and assigning them to the same global node in the mesh. Thus, the quadrilateral is identical to a three-noded triangle when the parameter f is unity. The limit value for the row sum method will give equal lumped terms at the three nodes while method (I.6) yields a lumped value for the coalesced node which is two-thirds the value at the other nodes and the quadrature method (I.9) yields a zero lumped value at the coalesced node. Thus, methods (I.6) and(I.9) give limit cases which depend on how the nodes are numbered to form each triangular element. This lack of invariance is not desirable in computer programs; hence for the four-noded quadrilateral, method (I.5) appears to be superior to the other two. On the other hand,we have observed above that the row sum method (I.5) leads to negative diagonal elements for the eight-noded element; hence there is no universal method for diagonalizing a matrix.

A fourth but not widely used method is available which may be explored to deduce a consistent matrix that is diagonal. This consists of making a mixed representation for the term creating the **A** matrix.

Consider a functional given by

$$\Pi_1 = \tfrac{1}{2}\int_\Omega \mathbf{u}^{\mathrm{T}}\mathbf{c}\mathbf{u}\,\mathrm{d}\Omega \tag{I.14}$$

The first variation of Π_1 yields

$$\delta\Pi_1 = \int_\Omega \delta\mathbf{u}^{\mathrm{T}}\mathbf{c}\mathbf{u}\,\mathrm{d}\Omega \tag{I.15}$$

Approximation using the standard form

$$\mathbf{u} \approx \hat{\mathbf{u}} = \mathbf{N}_i\tilde{\mathbf{u}}_i = \mathbf{N}\tilde{\mathbf{u}} \tag{I.16}$$

yields

$$\delta\Pi_1 = \delta\tilde{\mathbf{u}}^{\mathrm{T}}\int_\Omega \mathbf{N}^{\mathrm{T}}\mathbf{c}\mathbf{N}\,\mathrm{d}\Omega\,\tilde{\mathbf{u}} \tag{I.17}$$

This yields exactly the form for **A** given by Eq. (I.1).

We can construct an alternative mixed form by introducing a momentum type variable given by

$$\mathbf{p} = \mathbf{c}\mathbf{u} \tag{I.18}$$

The Hellinger–Reissner type mixed form may then be expressed as

$$\Pi_2 = \int_\Omega \mathbf{u}^{\mathrm{T}}\mathbf{p}\,\mathrm{d}\Omega - \tfrac{1}{2}\int_\Omega \mathbf{p}^{\mathrm{T}}\mathbf{c}^{-1}\mathbf{p}\,\mathrm{d}\Omega \tag{I.19}$$

and has the first variation

$$\delta\Pi_2 = \int_\Omega \delta\mathbf{u}^{\mathrm{T}}\mathbf{p}\,\mathrm{d}\Omega + \int_\Omega \delta\mathbf{p}^{\mathrm{T}}\left(\mathbf{u} - \mathbf{c}^{-1}\mathbf{p}\right)\mathrm{d}\Omega \tag{I.20}$$

The term with variation on $\mathbf{u}$ will combine with other terms so is not set to zero; however the other term will not appear elsewhere so can be solved separately.

If we now introduce an approximation for $\mathbf{p}$ as

$$\mathbf{p} \approx \hat{\mathbf{p}} = \mathbf{n}_j \tilde{\mathbf{p}}_j = \mathbf{n}\tilde{\mathbf{p}} \tag{I.21}$$

then the variational equation becomes

$$\delta\Pi_2 = \delta\tilde{\mathbf{u}}^{\mathrm{T}} \int_\Omega \mathbf{N}^{\mathrm{T}}\mathbf{n}\,\mathrm{d}\Omega\,\tilde{\mathbf{p}} + \delta\tilde{\mathbf{p}}^{\mathrm{T}}\left(\int_\Omega \mathbf{n}^{\mathrm{T}}\mathbf{N}\,\mathrm{d}\Omega\,\tilde{\mathbf{u}} - \int_\Omega \mathbf{n}^{\mathrm{T}}\mathbf{c}^{-1}\mathbf{n}\,\mathrm{d}\Omega\,\tilde{\mathbf{p}}\right) \tag{I.22}$$

If we now define the matrices

$$\begin{aligned} \mathbf{G} &= \int_\Omega \mathbf{N}^{\mathrm{T}}\mathbf{n}\,\mathrm{d}\Omega \\ \mathbf{H} &= \int_\Omega \mathbf{n}^{\mathrm{T}}\mathbf{c}^{-1}\mathbf{n}\,\mathrm{d}\Omega \end{aligned} \tag{I.23}$$

then the weak form is

$$\delta\Pi_2 = \begin{bmatrix}\delta\tilde{\mathbf{u}}^{\mathrm{T}} & \delta\tilde{\mathbf{p}}^{\mathrm{T}}\end{bmatrix}\left(\begin{bmatrix}\bullet & \mathbf{G}^{\mathrm{T}} \\ \mathbf{G} & -\mathbf{H}\end{bmatrix}\begin{Bmatrix}\tilde{\mathbf{u}} \\ \tilde{\mathbf{p}}\end{Bmatrix} = \begin{Bmatrix}\bullet \\ \mathbf{0}\end{Bmatrix}\right) \tag{I.24}$$

Eliminating $\tilde{\mathbf{p}}$ using the second row of Eq. (I.24) gives

$$\mathbf{A} = \mathbf{G}^{\mathrm{T}}\mathbf{H}^{-1}\mathbf{G} \tag{I.25}$$

for which diagonal forms may now be sought. This form again has the same options as discussed above but, in addition, forms for the shape functions $\mathbf{n}$ can be sought which also render the matrix diagonal.